APPROXIMATE SOLUTIONS OF GRETZ — NUSSELT PROBLEM

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- 1) ;
 - 2) (
 - 3) , -
 - 4) -
- [1],

$$\frac{1}{R} \frac{\partial}{\partial R} \left(R \frac{\partial \Theta(X, R)}{\partial R} \right) = (1 - R^2) \frac{\partial \Theta(X, R)}{\partial X}, \quad (1)$$

$$\Theta = \frac{g}{g_0} = \frac{t - t_0}{t_0 - t}, \quad R = \frac{r}{r_0}, \quad X = \frac{a}{2\pi r_0} \frac{x}{r_0} = \frac{2}{Pe} \frac{x}{d};$$

$$\begin{aligned} X = 0, \quad 0 \leq R < 1 & \quad \Theta = 1, \\ X \geq 0, \quad R = 0 & \quad \partial \Theta / \partial R = 0, \end{aligned} \quad (2)$$

$$X \geq 0, \quad R=1 \qquad \Theta = 0. \qquad (1), (2), (3) \qquad [1, 2] \qquad (3)$$

$$\Theta(X, R) = \sum_{N=0}^{\infty} A_n \exp(-\varepsilon_n^2 X) \Psi_n(r), \qquad (4)$$

$$\Psi_n(R) = \Psi(\varepsilon_n, \varepsilon_n R);$$

$$\Psi(\varepsilon R) = \sum_{n=0}^{\infty} b_{2n}(\varepsilon R)^{2n}, \qquad (5)$$

$$b_0 = 1, \quad n=0; \quad b_2 = -b_0/4 = -1/4, \quad n=1;$$

$$b_{2n} = \frac{1}{(2n)^2} \left(\frac{1}{\varepsilon^2} b_{2n-4} - b_{2n-2} \right), \quad n \geq 2;$$

$$A_n = -\frac{2}{\varepsilon_n} \left(\frac{\partial \Psi}{\partial \varepsilon} \right) \Big|_{\varepsilon=\varepsilon_n, R=1}. \qquad (6)$$

$$\frac{t-t_{ct}}{t_0-t_{ct}} = \sum_{n=0}^{\infty} A_n \exp(-2\varepsilon_n^2 \frac{x}{Ped}) \Psi_n(\frac{r}{r_0}). \qquad (7)$$

1 ε_n, A_n , -

.

[1](#)

ε_n

n	ε_n	A_n
0	2,704364	1,4764354
1	6,679031	-0,8061239
2	10,67338	0,5887621
3	14,671078	-0,4758504
4	18,669872	0,4050218
5	22,669143	-0,3557565
6	26,668662	0,319169
7	30,668323	-0,2907358
8	34,668074	0,2678911
9	38,667883	-0,2490625
10	42,667734	0,2332277

[1, 2], « n

ε_n », :

$$\varepsilon_n = 4n + 8/3, \qquad (8)$$

$$A_n = (-1)^n * 2,84606 \varepsilon_n^{-2/3}, \qquad (9)$$

(5), n . -

(3) -

$\Delta = 0,001 \div 1,0$ $\delta = 10^{-7} \div 10^{-3}$ -

$n = 6$, , (5, 6)

1. n (8, 9) 6.

$$(4) \quad (1),$$

$$\delta\Theta(X, R)$$

():

$$\delta\Theta(X, R) = \sum_{n=0}^{\infty} A_n \exp(-\varepsilon_n^2 X) \Psi_n''(R) + \frac{1}{R} \sum_{n=0}^{\infty} A_n \exp(-\varepsilon_n^2 X) \Psi_n'(R) - (1 - R^2) \sum_{n=0}^{\infty} A_n (-\varepsilon_n^2) \exp(-\varepsilon_n^2 X) \Psi_n(R). \quad (10)$$

$$X \quad R \quad (0 \leq X < \infty, 0 \leq R \leq 1)$$

$$n. \quad , \quad n \quad 3 \quad 4$$

$$\delta\Theta(X, R)|_{n=4} \quad (10)$$

		R										
		0	0,1	0,2	0,3	0,4	0,5	0,6	0,7	0,8	0,9	1
X	1E-05	0,00	0,17	41,76	1,0E+03	9,5E+03	5,2E+04	2,0E+05	5,6E+05	1,3E+06	2,2E+06	2,3E+06
	0,0001	0,00	0,16	40,41	9,9E+02	9,2E+03	5,0E+04	1,9E+05	5,4E+05	1,2E+06	2,1E+06	2,2E+06
	0,001	0,00	0,12	29,06	7,1E+02	6,6E+03	3,6E+04	1,4E+05	3,9E+05	8,8E+05	1,5E+06	1,6E+06
	0,01	0,00	0,00	0,89	21,88	2,0E+02	1,1E+03	4,2E+03	1,2E+04	2,6E+04	4,5E+04	4,3E+04
	0,1	0,00	0,00	0,00	0,00	-0,03	-0,14	-0,58	-1,81	-4,66	-10,24	-19,32
	1	0,00	0,00	0,00	0,00	0,00	0,00	0,00	0,00	0,00	0,00	0,00
	10	0,00	0,00	0,00	0,00	0,00	0,00	0,00	0,00	0,00	0,00	0,00
	100	0,00	0,00	0,00	0,00	0,00	0,00	0,00	0,00	0,00	0,00	0,00
	1000	0,00	0,00	0,00	0,00	0,00	0,00	0,00	0,00	0,00	0,00	0,00
	10000	0,00	0,00	0,00	0,00	0,00	0,00	0,00	0,00	0,00	0,00	0,00
	100000	0,00	0,00	0,00	0,00	0,00	0,00	0,00	0,00	0,00	0,00	0,00

$$, \quad 1E-05 = 1 \cdot 10^{-5}; \quad 2,3E+06 = 2,3 \cdot 10^{+6}.$$

$$\delta\Theta(X, R)|_{n=4} \quad (),$$

$$0 \leq R < 0,1; \quad 0 \leq X < \infty \quad 1 \leq X < \infty; \quad 0 \leq R \leq 1.$$

$$(1) \quad :$$

$$\Theta(X, R) = \Phi(X) \Psi(R). \quad (11)$$

(4) (1)

$$\Phi \Psi'' + \frac{1}{R} \Phi \Psi' = (1 - R^2) \Psi \Phi'.$$

$$\frac{d\Phi}{dX} = -\varepsilon^2 \Phi. \quad (12)$$

$$\Phi(X) = A_0 \exp(-\varepsilon^2 X). \quad (13)$$

$$\frac{d^2\Psi}{dR^2} + \frac{1}{R} \frac{d\Psi}{dR} + \varepsilon^2 (1 - R^2) \Psi = 0. \quad (14)$$

1) $R=1:$

$$\frac{d^2\Psi}{dR^2} + \frac{1}{R} \frac{d\Psi}{dR} = 0, \quad \frac{dU}{dR} = -\frac{U}{R}, \quad \frac{dU}{U} = -\frac{dR}{R}, \quad \ln(UR) = \ln C; \quad \frac{d\Psi}{dR} = U = \frac{C_1}{R};$$

$$\Psi(R) = C_1 \ln R + C_2; \quad (14)$$

2) $R=0:$

$$\frac{d^2\Psi}{dR^2} + \frac{1}{R} \frac{d\Psi}{dR} + \varepsilon^2 \Psi = 0, \quad 0/0$$

$$2 \frac{d^2\Psi}{dR^2} + \varepsilon^2 \Psi = 0 \quad \frac{d^2\Psi}{dR^2} + \frac{\varepsilon^2}{2} \Psi = 0. \quad (15)$$

$$\Psi(R) = A \cos(\mu R) + B \sin(\mu R), \quad \mu = \frac{\varepsilon}{\sqrt{2}}.$$

$$A \quad B \quad : \quad R=0 \quad \frac{d\Psi}{dR} = 0; \quad R=1 \quad \Psi(1) = 0.$$

$$, \quad B=0; \quad , \quad -$$

$$, \quad A=0 \quad \cos(\mu) = 0, \quad \mu_n = (2n-1) \frac{\pi}{2}.$$

$$\Psi(R) = \sum_{n=1}^{\infty} A_n \cos(\mu_n R). \quad (16)$$

$$(6) \quad (9) \quad (4),$$

$$\Theta(X, R) = \sum_{n=1}^{\infty} D_n \cos(\mu_n R) \exp(-\varepsilon_n^2 X), \quad (17)$$

$$D_n = A_0 A_n. \quad (10) \quad (2)$$

$$1 = \sum_{n=1}^{\infty} D_n \cos(\mu_n R).$$

[3],

$$\mu_n.$$

$$\int_0^1 \cos(\mu_n R) dR = D_n \int_0^1 \cos^2(\mu_n R) dR \quad D_n = \frac{\int_0^1 \cos(\mu_n R) dR}{\int_0^1 \cos^2(\mu_n R) dR}$$

$$D_n = \frac{\int_0^1 \cos(\mu_n R) dR}{\int_0^1 \cos^2(\mu_n R) dR} = \frac{2 \sin \mu_n}{\mu_n}.$$

$$D_n \quad (17),$$

(1)–(3).

$$\Theta(X, R) = \sum_{n=1}^{\infty} \frac{2 \sin \mu_n}{\mu_n} \cos(\mu_n R) \exp(-2 \mu_n^2 X). \quad (18)$$

(18)

$$R = 0.$$

(18)

$$\Theta_1(X, R).$$

$$0 \leq R < 1, 0 < X < \infty.$$

(18)

$$\frac{\partial \Theta(X, R)}{\partial X}, \frac{\partial \Theta(X, R)}{\partial R}, \frac{\partial^2 \Theta(X, R)}{\partial R^2}$$

(1)

n

$$\begin{aligned} \delta \Theta_1(X, R)|_{n=4} &= \frac{\partial^2 \Theta}{\partial R^2} + \frac{1}{R} \frac{\partial \Theta}{\partial R} - (1 - R^2) \frac{\partial \Theta}{\partial X} = \\ &= (2/R) \left[\begin{aligned} &\sin(\pi R/2) \exp(-\pi^2 X/2) - \sin(3\pi R/2) \exp(-9\pi^2 X/2) + \sin(5\pi R/2) \exp(-25\pi^2 X/2) - \\ &-\sin(7\pi R/2) \exp(-49\pi^2 X/2) \end{aligned} \right] + (19) \\ &+ \pi(1 - 2R^2) \left[\begin{aligned} &\cos(\pi R/2) \exp(-\pi^2 X/2) - 3\cos(3\pi R/2) \exp(-9\pi^2 X/2) + \\ &+ 5\cos(5\pi R/2) \exp(-25\pi^2 X/2) - 7\cos(7\pi R/2) \exp(-49\pi^2 X/2) \end{aligned} \right]. \end{aligned}$$

$$\partial\Theta_1(X, R)|_{n=4} \quad (19)$$

		R										
		0	0,1	0,2	0,3	0,4	0,5	0,6	0,7	0,8	0,9	1
X	1E-05	-25,057	-13,674	6,4381	11,951	0,5788	-8,8639	-5,5061	-1,4728	-9,3877	-15,204	7,9917
	0,0001	-24,387	-13,291	6,2962	11,623	0,5042	-8,669	-5,3288	-1,3945	-9,1989	-14,904	7,9179
	0,001	-18,544	-9,9709	5,0209	8,7609	-0,0894	-6,8917	-3,7402	-0,6921	-7,4943	-12,208	7,2416
	0,01	-0,879	-0,4006	0,351	0,4672	0,3876	1,1909	2,4239	2,0752	-0,278	-1,1727	3,9471
	0,1	3,614	3,5627	3,4094	3,1572	2,8166	2,4107	1,9794	1,579	1,2777	1,146	1,2446
	1	0,0452	0,0444	0,042	0,0383	0,0336	0,0283	0,0231	0,0185	0,0151	0,0136	0,0144
	10	2E-21	2E-21	2E-21	2E-21	2E-21	1E-21	1E-21	1E-21	8E-22	7E-22	7E-22
	100	3E-214	3E-214	3E-214	3E-214	2E-214	2E-214	2E-214	1E-214	1E-214	9E-215	1E-214
	1000	0	0	0	0	0	0	0	0	0	0	0
	10000	0	0	0	0	0	0	0	0	0	0	0
	100000	0	0	0	0	0	0	0	0	0	0	0

. 3

(19):

$$0 \leq R < 1, \quad 0,01 < X < \infty.$$

(18)

 Θ [1]:

$$\Theta = \frac{\bar{t} - t}{t_0 - t} = 2 \int_0^1 \Theta W_x R dR = 4 \int_0^1 \Theta (1 - R^2) R dR, \quad (20)$$

 Θ_1 Θ_1

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 $X \rightarrow \infty \quad \Theta_1 \rightarrow 0$ n ; $X \rightarrow 0 \quad \Theta_1 \approx 1.$ Θ_1 n

(20)

		X										
		1E-05	0,0001	0,001	0,01	0,1	1	10	100	1000	10000	100000
n	1	0,8009	0,8005	0,7969	0,7623	0,4889	0,0058	3E-22	0	0	0	0
	2	0,9952	0,9941	0,9829	0,887	0,4912	0,0058	3E-22	0	0	0	0
	3	0,9897	0,9886	0,9781	0,8854	0,4912	0,0058	3E-22	0	0	0	0
	4	0,9993	0,998	0,9856	0,8863	0,4912	0,0058	3E-22	0	0	0	0
	5	0,9976	0,9964	0,9845	0,8863	0,4912	0,0058	3E-22	0	0	0	0
	6	0,9997	0,9984	0,9856	0,8863	0,4912	0,0058	3E-22	0	0	0	0

(20)

 R ($R=1$)

(18)

[1]:

$$Nu = \frac{\alpha d}{\lambda} = \frac{-2}{\Theta} \left(\frac{\partial \Theta}{\partial R} \right) \Big|_{R=1}, \quad (21)$$

 α —, λ —

,

 d —

(—).

 Θ_1 $\left(\frac{\partial \Theta}{\partial R} \right) \Big|_{R=1},$ $X,$ n ,

,

 $n = 4$ $Nu(X).$

$$Nu_1(X) \quad n = 4, \quad (21)$$

	X										
	1E-05	0,0001	0,001	0,01	0,1	1	10	100	1000	10000	100000
n=4	256,03	255,01	245,18	178,59	88,252	79,911	79,911	79,911	79,911	79,911	79,911

$$Nu_1(X) \gg 1, \quad .5,$$

(18)

(14):

$$\frac{d^2\Psi}{dR^2} + \frac{1}{R} \frac{d\Psi}{dR} + \varepsilon^2(1-R^2)\Psi = 0,$$

$$\frac{d\Psi(R=0)}{dR} = 0, \quad \Psi(R=1) = 0.$$

$\Psi(R)$

$$\Psi(R) = C_1 + C_2(1-R^2).$$

1 2

$$\frac{d\Psi}{dR} = -2C_2R;$$

$$R=0 \quad \frac{d\Psi}{dR} = 0, \quad R=1 \quad 0 = C_1;$$

$$\Psi(R) = C_2(1-R^2). \quad (22)$$

$$\frac{d^2\Psi}{dR^2} = -1$$

$$[1]), \quad -1 = -2C_2, \quad C_2 = \frac{1}{2}.$$

(22)

$$\Psi(R) = \frac{1}{2}(1-R^2). \quad (23)$$

$\Psi(R)$ (14)

$$-1 - 2C_2 + \frac{\varepsilon^2}{2}(1-R^2)(1-R^2) = 0,$$

ε^2

$$\varepsilon^2 = \frac{4}{(1-R^2)^2},$$

$$R \rightarrow 0 \quad \varepsilon \rightarrow 2, \quad R \rightarrow 1 \quad \varepsilon \rightarrow \infty.$$

(11),

(1)–(3)

$$\Theta(X, R) = \Psi(R)\Phi(X) = \frac{1}{2}(1 - R^2)\exp(-\varepsilon^2 X). \quad (24)$$

$$\frac{\partial \Theta(X, R)}{\partial X}, \quad \frac{\partial \Theta(X, R)}{\partial R},$$

$$\frac{\partial^2 \Theta(X, R)}{\partial R^2}$$

$$\Theta(X, R):$$

$$\frac{\partial \Theta(X, R)}{\partial X} = \frac{2 \exp(-4X / (R^2 - 1)^2)}{(R^2 - 1)},$$

$$\frac{\partial \Theta(X, R)}{\partial R} = -\frac{\exp(-4X / (R^2 - 1)^2)}{(R^2 - 1)^2} R(R^4 - 2R^2 + 8X + 1),$$

$$\frac{\partial^2 \Theta(X, R)}{\partial R^2} = -\frac{\exp(-4X / (R^2 - 1)^2)}{(R^2 - 1)^5} (R^{10} - 5R^8 - 8R^6 X + 10R^6 + 8R^4 X - 10R^4 + 128R^2 X^2 + 8R^2 X + 5R^2 - 8X - 1)$$

(1)

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$$\partial \Theta_2(X, R)$$

:

$$\partial \Theta_2(X, R) = \frac{(R^4 - 2R^2 - 8R^2 X + 1)}{\exp(4X / (R^2 - 1)^2)(R^2 - 1)^5}. \quad (25)$$

6

$$(25),$$

3 %.

6

$$\partial \Theta_2(X, R)$$

(25)

		R										
		0	0,1	0,2	0,3	0,4	0,5	0,6	0,7	0,8	0,9	1
X	1E-05	-2E-04	-2E-04	-2E-04	-2E-04	-3E-04	-4E-04	-6E-04	-0,001	-0,003	-0,023	-2,514
	0,0001	-0,002	-0,002	-0,002	-0,002	-0,003	-0,004	-0,006	-0,012	-0,034	-0,232	-16,96
	0,001	-0,016	-0,017	-0,018	-0,021	-0,027	-0,038	-0,061	-0,121	-0,34	-2,138	12877
	0,01	-0,167	-0,172	-0,188	-0,221	-0,281	-0,393	-0,626	-1,195	-2,825	56,162	1E+16
	0,1	-2,387	-2,46	-2,694	-3,142	-3,895	-4,977	-4,811	28,471	2215,9	3E+08	8E+117
	1	-873,6	-896,8	-905,8	-347,2	6366,9	118778	6E+06	8E+09	3E+17	6E+53	-

$$Nu_2$$

$$\Theta_2 \quad ($$

(20)

$$(\partial \Theta(X, R) / \partial R) \big|_{R=1} \quad ($$

(

)

$$R=1,$$

).

$$Nu_2$$

7.

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7,

$$Nu_2(X)$$

$$X \leq 0,001.$$

$$\Theta_2, (\partial \Theta(X, R) / \partial R) \big|_{R=1}, Nu_2(X), \quad (21)$$

		Θ_2	$(\partial \Theta(X, R) / \partial R) \big _{R=1}$	$Nu_2(X)$
X	1E-05	0,0001611	-1,076	13358,16263
	0,0001	0,001107	-1,089	1967,479675
	0,001	0,006717	-0,0008616	0,2565431
	0,01	0,029	0	0
	0,1	0,046	0	0
	1	0,000877	0	0
	5	3,541E-11	0	0
	10	0	0	-
	100	0	0	-

1. —, 1997.
2. —, 1967.
3. —, 1975

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